ELSEVIER
Journal of Computational and Applied Mathematics 140 (2002) 773–783
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Series solutions of coupled differential equations with one regular singular point

K.W. Tomantschger

Department of Mathematics, Technical University of Graz, Steyrergasse 30, A-8010 Graz, Austria

Received 5 September 2000; received in revised form 10 October 2001

Abstract

We consider two linear second-order ordinary differential equations. $r = 0$ is a regular singular point of these equations. Applying the *classical Method of Frobenius*, we do not obtain any indicial equation and therefore no solution, because the differential equations are coupled.

In this paper, we present an *extended Method of Frobenius* on a coupled system of two ordinary differential equations. These equations come from the micropolar theory, which is one of the three kinds of the new 3M physics. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 34A30; 34A25; 33C10; 34B05

Keywords: Coupled ordinary differential equations; Regular singular point; Series solutions in closed form; Extended Frobenius' method; Indicial equations; Modified Bessel functions; Boundary conditions; Micropolar theory

1. Introduction

In the classical continuum mechanics, the stress states are described by means of a symmetric stress tensor. Since the classical continuum model is not sufficient for the description of the behavior of certain materials, e.g., granular materials, fluid suspensions, liquid crystals, blood flow, etc., the continuum model with microstructure has been introduced [5].

Eringen and Suhubi [6] introduced micropolar continuum and micropolar fluid models, respectively, characterized by the couple stress and a nonsymmetric stress tensor. This theory comprises of two independent kinematic quantities: the velocity vector v_i and the spin or microrotation vector v_i.

E-mail address: toman@weyl.math.tu-graz.ac.at (K.W. Tomantschger).

PII: S0377-0427(01)00598-2

Micropolar fluid can, among other applications, be used in describing the behavior of motion of suspensions as a mixture of two phases [1–3,7,8]. The basic phase of the suspension is a fluid, whereas the dispersive phase consists of solid particles.

Let us consider the following experiment of the *micropolar theory* [4].

A suspension has the stationary motion between two coaxial cylinders. The outer cylinder with the radius R_2 rotates with angular velocity Ω, while the inner with the radius R_1 does not move. The axis of rotation is horizontal.

The behavior of the micropolar suspension between the two coaxial cylinders is described by the system of the two coupled differential equations

$$(\mu + k)[r^2 \upsilon'' + r\upsilon' - \upsilon] - kr^2 v' = 0, \tag{1}$$

$$\gamma[rv'' + v'] + k(\upsilon r)' - 2krv = 0. \tag{2}$$

$\upsilon(r)$ is the unknown velocity of the suspension (macromotion) and $v(r)$ represents the unknown microrotational velocity. r is one of the coordinates of the cylindrical system. $\gamma, \mu, k \in \mathbb{R}_0^+$ denote the viscosity coefficients of the micropolar continuum.

Treating (1) resp. (2) as a nonhomogeneous second-order differential equation in $\upsilon(r)$ resp. $v(r)$, we see, that r=0 is a regular singular point of (1) and (2). But the Method of Frobenius, evaluated for *one* nth-order ordinary differential equation with *one* unknown function, is not applicable, because our system is coupled. Since, if two coupled differential equations are given, we do not know which terms (implying the lowest power of the series' variable) of these equations yield an indicial equation. These terms also can be in both equations. After solving this problem and finding the indicial equations in Section 2, solutions will be constructed by the recurrence relations for a smaller root and another one of the indicial equations in Section 3. Finally, a fundamental system of Eqs. (1) and (2) is given in Section 4. The entire solution can also be represented in closed form. The velocity $\upsilon(r)$ and the microrotational velocity $v(r)$ are represented by modified Bessel functions of the first kind, by MacDonald functions and by powers of r. In Section 5, the arbitrary constants of the general solution are computed by boundary conditions. Finally, in the last Section 6 it can be shown that this obtained special solution can be reduced to the classical case of suspension motion, which we get by treating this technical problem with the laws of the classical phsysics.

2. The indicial equations of a coupled system

Since $r = 0$ is a regular singular point of the differential equations, (1) and (2) have *at least one* nontrivial series solution of the form

$$\upsilon(r) = (\beta r)^{\lambda} \sum_{n=0}^{\infty} a_n (\beta r)^n, \quad a_0 \neq 0 \tag{3}$$

and

$$v(r) = (\beta r)^{\varrho} \sum_{n=0}^{\infty} b_n (\beta r)^n, \quad b_0 \neq 0. \tag{4}$$

λ and ϱ are definite (real or complex) constants which must be determined. They are roots of indicial equations which still must be derived. β is an undetermined real constant and will be fixed later.

We only need solutions for $R_1 \leqslant r \leqslant R_2$, $R_1 > 0$. This solution, (3) and (4), is valid in some deleted interval $0 < r < R$, $R > R_2$ (about 0).

Remark 1. Having only one ordinary differential equation

$$(x-x_0)^2 y'' + (x-x_0)P(x)y' + Q(x)y = 0, \quad y = y(x), \tag{5}$$

the indicial equation can be obtained as the factor of a_0 and the lowest power of the independent variable (if we set $y = (x-x_0)^\lambda[a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots])$.

Substituting (3) and (4) and their derivatives into (1) resp. (2), the lowest powers of r of the υ-series resp. v-series can appear as well in (1) resp. in (2) as in (1) and (2). So it is suitable to add Eqs. (1) and (2) and to separate υ and v derivatives. On the left-hand side only stand the υ derivatives, on the right-hand side is the new v equation. This relation can be seen as an *inhomogeneous* second-order differential equation in $\upsilon(r)$ or $v(r)$. A very important point is that in Eqs. (1) and (2) the factors of υ'' and v'' are the same, for example r^2 or 1. Changing the factor of e.g. v'' implies the addition of other υ and υ' terms (see $(\upsilon r)'$) to (1). And in the same way $-kr^2v'$ of (1) obtains another summand. We get different indicial equations resp. different λ and ϱ values.

In our case, we divide (1) by $\mu + k$ and multiply (2) by $r\gamma^{-1}$. Adding these two equations and separating υ and v derivatives yields

$$r^2\upsilon'' + r\left(-\frac{k}{\gamma}r + 1\right)\upsilon' - \left(\frac{k}{\gamma}r + 1\right)\upsilon = r^2v'' + r\left(\frac{k}{\mu+k}r + 1\right)v' - \frac{2k}{\gamma}r^2v. \tag{6}$$

Differentiating the infinite power series in (3) and (4) term by term twice, we obtain

$$\upsilon' = \beta\sum_{n=0}^{\infty}(n+\lambda)a_n(\beta r)^{n+\lambda-1}, \quad \upsilon'' = \beta^2\sum_{n=0}^{\infty}(n+\lambda)(n+\lambda-1)a_n(\beta r)^{n+\lambda-2},$$

$$v' = \beta\sum_{n=0}^{\infty}(n+\varrho)b_n(\beta r)^{n+\varrho-1}, \quad v'' = \beta^2\sum_{n=0}^{\infty}(n+\varrho)(n+\varrho-1)b_n(\beta r)^{n+\varrho-2}.$$

Remark 2. The following generally holds.

(I) Let

$$F(x) = \sum_{j=0}^{\infty} f_j(x) = f_0(x) + f_1(x) + f_2(x) + \cdots$$

be a convergent series in the interval $|x-a| < r_1$. Furthermore, we suppose that

$$\sum_{j=0}^{\infty} f_j'(x) = f_0'(x) + f_1'(x) + f_2'(x) + \cdots$$

is a uniform convergent series of continuous functions $f_j'(x)$ in $|x-a| < r_1$.

Then

$$F'(x) = f_0'(x) + f_1'(x) + f_2'(x) + \cdots .$$

(II) A power series with a nonzero radius of convergence R is uniformely convergent in every circular disk $|x-a| \leqslant r_1$ of radius $r_1 < R$.

These two statements allow us to differentiate a convergent power series term by term.

Inserting (3) and (4) and their derivatives into (6), then splitting off the first term of every series, we find

$$
\begin{aligned}
&(\lambda^2-1)a_0(\beta r)^{\lambda}+\sum_{n=1}^{\infty}[(n+\lambda)^2-1]a_n(\beta r)^{n+\lambda}\\
&\quad-\frac{k}{\gamma\beta}(1+\lambda)a_0(\beta r)^{\lambda+1}-\frac{k}{\gamma\beta}\sum_{n=2}^{\infty}(n+\lambda)a_{n-1}(\beta r)^{n+\lambda}\\
&=\varrho^2 b_0(\beta r)^{\varrho}+\sum_{n=1}^{\infty}(n+\varrho)^2 b_n(\beta r)^{n+\varrho}\\
&\quad+\frac{k}{(\mu+k)\beta}\varrho b_0(\beta r)^{\varrho+1}+\frac{k}{(\mu+k)\beta}\sum_{n=2}^{\infty}(n+\varrho-1)\,b_{n-1}(\beta r)^{n+\varrho}\\
&\quad-\frac{2k}{\gamma\beta^2}b_0(\beta r)^{\varrho+2}-\frac{2k}{\gamma\beta^2}\sum_{n=3}^{\infty}b_{n-2}(\beta r)^{n+\varrho}.
\end{aligned}
\tag{7}
$$

Treating (6) as an inhomogeneous differential equation in υ, the right-hand side of (7) is the perturbation function of (6) represented by power series. Since, by assumption $a_0 \neq 0$, the coefficient of the lowest power of βr on the left-hand side of (7) will be equated to zero. In this way we find the *indicial equation*

$$\lambda^2-1=0$$

for the υ-series. Analogically, we obtain the *indicial equation* for the v-series

$$\varrho^2=0.$$

The roots of these two indicial equations are

$$\lambda_1=1,\quad \lambda_2=-1 \quad\text{and}\quad \varrho={}_1\varrho_2=0. \tag{8}$$

3. Computation of the coefficients a_n and b_n

The recurrence formulas we obtain by substituting the power series of $\upsilon(r), v(r)$ and their derivatives into the differential Eqs. (1) and (2). Simplifying the resulting relations, we write (1) as

$$\sum_{n=0}^{\infty}[(n+\lambda)^2-1]a_n(\beta r)^{n+\lambda}=\frac{k}{\mu+k}\frac{1}{\beta}\sum_{n=2}^{\infty}(n+\varrho-2)b_{n-2}(\beta r)^{n+\varrho-1}. \tag{9}$$

With (2) we find

$$\beta\sum_{n=0}^{\infty}(n+\varrho)^2 b_n(\beta r)^{n+\varrho-1}-\frac{2k}{\gamma}\frac{1}{\beta}\sum_{n=2}^{\infty}b_{n-2}(\beta r)^{n+\varrho-1}=-\frac{k}{\gamma}\sum_{n=0}^{\infty}(n+\lambda+1)a_n(\beta r)^{n+\lambda}. \tag{10}$$

$\lambda_1-\lambda_2=2$ is a positive integer. A series solution in the form of (3) and (4), without any logarithm, surely exists if we take $\lambda_1=1$ and $\varrho=0$, because $\lambda_1>\lambda_2$.

3.1. Particular solution corresponding to $\varrho = 0$ and the smaller root $\lambda_2 = -1$

Solving the ordinary differential Eq. (5), we can observe that if the difference $\lambda_1 - \lambda_2$ between the roots of the indicial equation is a positive integer, it is sometimes possible to obtain the general solution using the smaller one, without bothering to find explicitly the solution corresponding to the larger one. So we use $\lambda_2 = -1$ and $\varrho = 0$. Inserting these indices into (9) and (10) we get by comparison of coefficients

$$a_1 = 0, \quad n(n-2)a_n = \frac{k}{\mu + k}\frac{1}{\beta}(n-2)b_{n-2}, \;\; n \geqslant 2, \tag{11}$$

and

$$b_1 = -\frac{k}{\gamma}\frac{1}{\beta}a_1, \quad \frac{k}{\gamma}na_n = \frac{2k}{\gamma}\frac{1}{\beta}b_{n-2} - \beta n^2 b_n, \;\; n \geqslant 2. \tag{12}$$

For eliminating a_n we multiply (12) by $-k^{-1}\gamma(n-2)$ and add the recurrence relations (11) and (12). This yields the recurrence formula

$$(n-2)n^2 b_n - \frac{1}{\beta^2}\frac{k}{\gamma}\left(\frac{2\mu + k}{\mu + k}\right)(n-2)b_{n-2} = 0, \quad n \geqslant 2.$$

Now we define

$$\frac{1}{\beta^2}\frac{k}{\gamma}\left(\frac{2\mu + k}{\mu + k}\right) := 1 \quad \text{resp. } \beta := \left[\frac{k}{\gamma}\left(\frac{2\mu + k}{\mu + k}\right)\right]^{1/2}. \tag{13}$$

Thus,

$$(n-2)n^2 b_n = (n-2)b_{n-2}, \quad b_n = b_n(\lambda_2, \varrho), \quad n \geqslant 2. \tag{14}$$

Since $a_1 = 0$, the first relation in (12) implies $b_1 = 0$. With the above recursion formula we find successively from this that for n odd

$$b_3 = b_5 = b_7 = \cdots = b_{2j+1} = \cdots = 0.$$

Letting $n = 2$ in (14), we obtain with $b_0 \neq 0$

$$0 \cdot b_2 = 0$$

and hence

$$b_2 \cdots \text{arbitrary}.$$

Thus b_2 is independent of the arbitrary constant b_0. It is a second arbitrary constant. From now on we can write

$$b_n = \frac{1}{n^2}b_{n-2}, \quad n \geqslant 4$$

instead of formula (14). For n even,

$$b_4 = \frac{1}{4^2}b_2, \quad b_6 = \frac{1}{6^2}b_4 = \frac{1}{4^2 6^2}b_2$$

and in general

$$b_{2j} = \frac{1}{4^2 6^2 8^2 \cdots (2j)^2}b_2, \;\; j \geqslant 2 \quad \text{resp. } b_{2j} = \frac{1}{2^{2j-2}(j!)^2}b_2, \;\; j \geqslant 1.$$

Substituting these coefficients into the power series of $v(r)$ in (4) and adding $4b_2 - 4b_2$, we get the solution $v = v_1(r)$

$$v_1(r) = b_0 - 4b_2 + 4b_2 \sum_{j=0}^{\infty} \frac{1}{j!j!} \left(\frac{\beta^2 r^2}{4} \right)^j = b_0 - 4b_2 + 4b_2 I_0(\beta r). \tag{15}$$

I_0 represents the *modified Bessel function of the first kind of order zero*.

Now we wish to find a homogeneous particular solution $v(r)$. Replacing n by $n+2$ in (11), expressing nb_n explicitly from the resulting relation and $(n-2)b_{n-2}$ from (11), putting these two terms into (12) which we multipled by $(n-2)$, we find by using (13),

$$(n-2)n^2(n+2)a_{n+2} = (n-2)na_n,$$
$$a_{-1} = 0, \quad n = -1, 0, 1, 2, \ldots \, . \tag{16}$$

Starting with $a_1 = 0$ from (11), it follows from this recurrence formula that

$$a_3 = a_5 = \cdots = a_{2j+1} = \cdots = 0.$$

Since $a_0 \neq 0$, (16) implies with $n = 0$ that

$$0 \cdot a_2 = 0.$$

This yields

$$a_2 \cdots \text{arbitrary}.$$

We obtain from (16) by taking $n = 2$

$$0 \cdot a_4 = 0$$

and, therefore,

$$a_4 \cdots \text{arbitrary}.$$

For the smaller root $\lambda_2 = -1$ and $\varrho = 0$ the recurrence relation (16) now becomes

$$a_{n+2} = \frac{1}{n(n+2)} a_n, \quad n \geqslant 4.$$

So

$$a_6 = \frac{1}{4 \cdot 6} a_4,$$

$$a_8 = \frac{1}{6 \cdot 8} a_6 = \frac{1}{4 \cdot 6^2 \cdot 8} a_4,$$

$$a_{10} = \frac{1}{8 \cdot 10} a_8 = \frac{1}{4 \cdot 6^2 \cdot 8^2 \cdot 10} a_4$$

and generally

$$a_{2j} = \frac{a_4}{4 \cdot 6^2 \cdot 8^2 \cdot 10^2 \cdots (2j-2)^2 \cdot 2j}, \; j \geqslant 4 \quad \text{resp. } a_{2j} = \frac{16 a_4}{2^{2j-1}(j-1)!j!}, \quad j \geqslant 2.$$

These coefficients imply the solution $v = v_1(r)$

$$v_1(r) = \frac{1}{\beta r}[a_0 + a_2(\beta r)^2] + 16a_4 \sum_{j=2}^{\infty} \frac{1}{(j-1)!j!} \left(\frac{\beta r}{2}\right)^{2j-1}.$$

Replacing $j-1$ by j and adding $16a_4\beta r/2 - 8a_4(\beta r)$, this solution may be written as

$$\begin{aligned} v_1(r) &= \frac{a_0}{\beta}\frac{1}{r} + (a_2 - 8a_4)\beta r + 16a_4 \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \left(\frac{\beta r}{2}\right)^{2j+1} \\ &= \frac{a_0}{\beta}\frac{1}{r} + (a_2 - 8a_4)\beta r + 16a_4 I_1(\beta r), \end{aligned} \tag{17}$$

where I_1 denotes the *modified Bessel function of the first kind of order one*.

4. Construction of a fundamental system of the differential Eqs. (1) and (2)

Now we try to express the arbitrary coefficients $a_2 - 8a_4$ and $16a_4$ in (17) by b_n, because in relation (15) b_0 and b_2 are used. Letting $n = 2$ and $n = 4$ in recurrence relation (12) we obtain with $b_4 = 16^{-1}b_2$ from (14)

$$a_2 - 8a_4 = \frac{1}{\beta}(b_0 - 4b_2) \quad \text{and} \quad 16a_4 = \frac{4}{\beta}\frac{k}{\mu + k}b_2.$$

Inserting these relations into (17) yields

$$v_1(r) = \frac{a_0}{\beta}\frac{1}{r} + (b_0 - 4b_2)r + \frac{4}{\beta}\frac{k}{\mu + k}b_2 I_1(\beta r). \tag{18}$$

Comparing this result with solution (15)

$$v_1(r) = b_0 - 4b_2 + 4b_2 I_0(\beta r),$$

we see, that we already have three arbitrary constants of integration which are independent of each other. Thus we define

$$C_1 = b_0 - 4b_2, \quad C_2 = \frac{a_0}{\beta}, \quad C_3 = 4b_2.$$

Now the particular solutions (18) (resp. (17)) and (15) become

$$v_1(r) = C_1 r + C_2\frac{1}{r} + \frac{1}{\beta}\frac{k}{\mu + k}C_3 I_1(\beta r) \tag{19}$$

and

$$v_1(r) = C_1 + C_3 I_0(\beta r). \tag{20}$$

But we still need a v solution which is linearly independent of (19) (resp. (17)). Furthermore, we have to find a v solution which is linearly independent of (20) (resp. (15)). These family of solutions must have the form

$$v_3(r) = CI_1(\beta r)\ln r + \sum_{n=0}^{\infty} A_n(\beta r)^{n+1}, \quad C \neq 0 \tag{21}$$

and

$$v_3(r) = I_0(\beta r)\ln r + \sum_{n=0}^{\infty} B_n(\beta r)^n. \tag{22}$$

Such a linearly independent solution $v_3(r)$ or $v_3(r)$ may be found by the method of Reduction of Order.

Remark 3. If we are correct, we have to write $\ln(\beta r)$ in (21) and (22). But

$$I_p(\beta r)\ln(\beta r) = I_p(\beta r)\ln\beta + I_p(\beta r)\ln r, \quad p = 0, 1.$$

The factor $I_p(\beta r)$ of the first additive term on the right-hand side of this relation represents a particular solution, already obtained in (15) (resp. (20)) and (17) (resp. (19)). So we may omit it.

Since I_1 in (19) and I_0 in (20) possess the same arbitrary constant C_3, they only differ by a factor, we also can expect this from (21) and (22), but with a new fourth arbitrary constant. The coupled system of second-order differential Eqs. (1) and (2) allows exactly four independent integration constants for the general solution.

Remark 4. We take the particular solutions

$$v_{1,1}(r) = C_1 \quad \text{and} \quad v_{1,1}(r) = C_1 r + C_2\frac{1}{r}.$$

Substituting the constant $v_{1,1}(r) = C_1$ into (1) we get the Euler–Cauchy equation

$$r^2 v'' + r v' - v = 0,$$

which is identically satisfied by $v_{1,1}(r)$. $v_{1,1}(r)$ and $v_{1,1}(r)$ satisfy Eq. (2) also identically.

Looking at the differential equations (1) (resp. (2)) we see that the first and second derivatives in the brackets are exactly the same as these of the modified Bessel equations of order one (resp. zero). So it is obvious that our second linearly independent solution (21) and (22) is represented by the *modified Bessel functions of the second kind*

$$v_3(r) = K_1(\beta r) = \left[\ln\left(\frac{\beta r}{2}\right) + \tilde{\gamma}\right] I_1(\beta r) + \frac{1}{\beta r} - \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!}\left(\frac{\beta r}{2}\right)^{2j+1}\left[\Phi(j) + \frac{1}{2(j+1)}\right] \tag{23}$$

and

$$v_3(r) = K_0(\beta r) = -\left[\ln\left(\frac{\beta r}{2}\right) + \tilde{\gamma}\right] I_0(\beta r) + \sum_{j=0}^{\infty} \frac{1}{j!j!}\left(\frac{\beta r}{2}\right)^{2j}\Phi(j). \tag{24}$$

Here $\Phi(j)$ is defined by

$$\Phi(j) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}, \quad \Phi(0) = 0.$$

$\tilde{\gamma}$ is the so-called *Euler–Mascheroni constant*, which is defined as the following limit:

$$\tilde{\gamma} = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right) = 0.5772156\ldots\,.$$

The K_n are also called *MacDonald functions*.

Let us assume a general solution of the form

$$v(r) = C_1 r + C_2 \frac{1}{r} + \frac{1}{\beta} \frac{k}{\mu + k} C_3 I_1(\beta r) + A C_4 K_1(\beta r) \tag{25}$$

and

$$\nu(r) = C_1 + C_3 I_0(\beta r) + C_4 K_0(\beta r). \tag{26}$$

We determine the arbitrary constant A by inserting the particular solution

$$v_{3,1} = A K_1(\beta r) \quad \text{and} \quad \nu_{3,1} = K_0(\beta r)$$

into the given differential equation (1). By using the relation $K_0'(x) = -K_1(x)$ we obtain

$$A = -\frac{1}{\beta} \frac{k}{\mu + k}.$$

Thus we may give the following result:

Proposition 5. *The velocity of the suspension* $v(r)$ *and the microrotational velocity* $\nu(r)$ *may be written as*

$$v(r) = C_1 r + C_2 \frac{1}{r} + \frac{1}{\beta} \frac{k}{\mu + k} \left[C_3 I_1(\beta r) - C_4 K_1(\beta r) \right], \quad r > 0 \tag{27}$$

and

$$\nu(r) = C_1 + C_3 I_0(\beta r) + C_4 K_0(\beta r), \quad r > 0 \tag{28}$$

with

$$\beta = \left[\frac{k}{\gamma} \left(\frac{2\mu + k}{\mu + k} \right) \right]^{1/2}.$$

These formulas express the general series solution (3) *and* (4)

$$v = (\beta r)^{\lambda} \sum_{n=0}^{\infty} a_n (\beta r)^n \quad \textit{and} \quad \nu = (\beta r)^{\varrho} \sum_{n=0}^{\infty} b_n (\beta r)^n$$

(*corresponding to* $\lambda_2 = 1$ *and* $\varrho = 0$) *and* (23) *and* (24)

$$v_3(r) = C I_1(\beta r) \ln r + \sum_{n=0}^{\infty} A_n (\beta r)^{n+1}, \quad C \neq 0,$$

$$\nu_3(r) = I_0(\beta r) \ln r + \sum_{n=0}^{\infty} B_n (\beta r)^n$$

(*corresponding to* $\lambda_1 = -1$ *and* $\varrho = 0$) *of the coupled system* (1) *and* (2)

$$(\mu + k)[r^2 v'' + r v' - v] - k r^2 \nu' = 0,$$

$$\gamma [r \nu'' + \nu'] + k (v r)' - 2 k r \nu = 0$$

in closed form. I_0 and I_1 represent the modified Bessel functions of the first kind of order zero resp. one. K_0 and K_1 are the modified Bessel functions of the second kind of order zero resp.one.

Proof. Our final solution, (27) and (28), satisfies the second-order system identically.

5. Boundary values

Let us consider the case of the motion of suspension when the suspension cleaves down the walls of the cylinders, so that the boundary conditions for the velocity $v(r)$ and the microrotational velocity $\nu(r)$ are

$$r = R_1: \quad v = 0, \quad \nu = 0, \tag{29}$$

$$r = R_2: \quad v = \Omega R_2, \quad \nu = 0. \tag{30}$$

Applying the boundary conditions to (27) and (28), we obtain an inhomogeneous linear system. Computing the constants we have

$$\begin{aligned}
C_1 &= \frac{\Omega R_2^2}{H}[I_0(a_2)K_0(a_1) - I_0(a_1)K_0(a_2)],\\
C_2 &= \frac{\Omega R_1 R_2^2}{H}[[I_1(a_1)(K_0(a_1) - K_0(a_2)) + K_1(a_1)(I_0(a_1) - I_0(a_2))]\\
&\quad + R_1(I_0(a_1)K_0(a_2) - I_0(a_2)K_0(a_1))],\\
C_3 &= \frac{\Omega R_2^2}{H}[K_0(a_2) - K_0(a_1)],\\
C_4 &= \frac{\Omega R_2^2}{H}[I_0(a_1) - I_0(a_2)], \quad a_j = \beta R_j, \ \ j = 1,2
\end{aligned}$$

with

$$\begin{aligned}
H = {} & (R_1^2 - R_2^2)(I_0(a_1)K_0(a_2) - I_0(a_2)K_0(a_1)) + [(K_0(a_1) - K_0(a_2))(R_1 I_1(a_1) - R_2 I_1(a_2))\\
& + (I_0(a_1) - I_0(a_2))(R_1 K_1(a_1) - R_2 K_1(a_2))].
\end{aligned}$$

6. Conclusion

The classical case of suspension motion is obtained if the material constant k of the micropolar suspension is equalized with zero. In that case, the relation for the velocity of suspension motion is reduced to the well-known form

$$v = \frac{\Omega}{R_2^2 - R_1^2}\left(R_2^2 r - \frac{R_1^2 R_2^2}{r}\right); \quad \nu = 0.$$

This velocity $v(r)$ we had gotten if we had solved this technical problem with laws of the classical physics.

References

[1] G. Ahmadi, A generalized continuum theory for multiphase suspension flows, Int. J. Eng. Sci. 23 (1985) 1.
[2] P. Cvetković, Continuum theory applied to suspension flows, XVII Yugoslav Meeting of Theoretical and Applied Mechanics, Zadar, 1986.
[3] P. Cvetković, On hydrodynamics of mixtures with nonsymmetric stress tensor—the case of suspension flow, Bull. Appl. Math. BAM 627 (1989).
[4] P. Cvetković, K.W. Tomantschger, D. Kuzmanović, Z. Golubović, The motion of a suspension between two coaxial cylinders, Facta Univ. (2001), in press.
[5] A.C. Eringen, Simple microfluids, Int. J. Eng. Sci. 2 (1964) 205.
[6] A.C. Eringen, E.S. Suhubi, Nonlinear theory of simple microelastic solids, Int. J. Eng. Sci. 2 (1964) 189.
[7] Z. Golubović, P. Cvetković, On the principle of virtual work in the theory of immiscible mixtures containing interface, Theoret. Appl. Mech. 15 (1989) 31.
[8] L.G. Petrosjan, Monography, University of Erevan, 1984.